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Perfect matchings of a graph associated with a binary de Bruijn digraph

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ABSTRACT

Let B_n be the binary de Bruijn digraph of order n and W the quotient set of the set of vertices of B_n with respect to the equivalence relation of rotation. Let G be the graph which has W as the set of vertices and in which two elements C and H are adjacent when there exist a vertex v of C and a vertex u of H such that (v, u) is an arc of B_n. Recently the problem of establishing whether the graph G has a perfect matching was posed. In this paper we answer in the positive to this problem in the case of n odd.

1. Introduction

For $n \geq 2$, let $\mathbb{Z}_2^n$ be the set of all n-ples $(a_1, a_2, \ldots, a_n)$ where $a_i \in \{0, 1\}$ and the subscripts are viewed modulo n. The binary de Bruijn digraph B_n has $\mathbb{Z}_2^n$ as the vertex set and two vertices $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are adjacent whenever

$$a_2, \ldots, a_n = b_1, \ldots, b_{n-1}.$$

It is well known that B_n is a strongly connected regular digraph of degree 2 and diameter n.

Let W be the quotient set of the set of vertices of B_n with respect to the equivalence relation of rotation and $G(B_n)$, or G for short, the graph which has W as the set of vertices and in which two elements C and H are adjacent when there exist a vertex v of C and a vertex u of H such that (v, u) is an arc of B_n. Recently the problem of establishing whether the graph G has a perfect matching was posed. In this paper we answer in the positive to this problem in the case of n odd, while the even case remains unsolved.

Recall that an ordered partition (or composition) of an integer n is a sequence $\lambda = (k_1, k_2, \ldots, k_m)$ where $k_i > 0$ and $k_1 + k_2 + \cdots + k_m = n$. The integers k_i are the parts of the partition. We may associate with λ a $(0, 1)$-sequence $s(\lambda)$ of length n, obtained from λ by replacing every part k_i with a sequence of $k_i - 1$ zeros and 1 one.

Conversely, given a $(0, 1)$-sequence $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, having $a_n = 1$, we associate with $\mathbf{a}$ an ordered partition of n, p($\mathbf{a}$), where the first part is the number of zeros before the first one plus 1, the second part is the number of zeros between the first and the second one plus 1 and so on. Thus $p(s(\lambda)) = \lambda$.

For example the partition $(4, 1, 2)$ of 7 may be represented by the string $(0, 0, 0, 1, 1, 0, 1)$ and conversely.

We may interpret $(0, 1)$-sequences in base 2 and associate with every $(0, 1)$-sequence $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ the number

$$n(\mathbf{a}) = a_n + a_{n-1}.2 + \cdots + a_1.2^{n-1}.$$

The set of $(0, 1)$-sequences of length n is ordered by the order relation which establishes $\mathbf{a} \leq \mathbf{b}$ if and only if $n(\mathbf{a}) \leq n(\mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ are $(0, 1)$-sequences of length n, ending with 1. A similar order relation holds for the set of ordered partitions of n, when we associate to a partition λ of n the number $n(\lambda) = n(s(\lambda))$.

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Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be a $(0, 1)$-sequence of length n. Given the permutation $\pi = (12\ldots n)$, the action of π^i on $\mathbf{a}$ is

$$\pi^i(\mathbf{a}) = (a_{1+i}, a_{2+i}, \ldots, a_i)$$

where $1 \le i \le n$ and the indices are modulo n. Under the action of the cyclic group C_n the orbits are determined and in each orbit we select the element whose numerical value is minimal; denote by $N(\mathbf{a})$ the minimum among the values obtained by rotation of $\mathbf{a}$.

In [1] it is proved that the number of orbits is

$$\frac{1}{n}\sum_{d|(n,m)} \binom{\frac{n}{d}}{\frac{m}{d}} \phi(d)$$

where (n, m) is the greatest common divisor of n and m and ϕ is the Euler function.
Moreover, given the composition $\lambda = (k_1, \ldots, k_m)$ and the permutation $\sigma = (12\ldots m)$, the action of σ^j on λ is $\sigma^j(\lambda) = (k_{1+j}, \ldots, k_{m+j})$ where $1 \le j \le m$ and the indices are mod m.

In relation to the partition λ, we define

$$N(\lambda) = N(s(\lambda)).$$

2. Partition of the vertices of B_n into cycles

Let $v_1 = (a_1, a_2, \ldots, a_n)$ be a vertex of B_n and A the circulant matrix

$$A := \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & a_1 & \cdots & a_{n-1} \end{pmatrix} \tag{1}$$

where every row, different from the first one, is obtained from the preceding by cyclically shifting every element one position to the left.
Denote by r the number of distinct rows of A; without loss of generality we may assume that the first r rows are distinct. Let v_j, $1 \le j \le r$ be the vertices corresponding to the first rows. It is easy to see that (v_j, v_{j+1}) is an arc of B_n, $1 \le j \le r-1$, as well as (v_r, v_1).
A consequence is that these vertices form a cycle $C_1 = (v_1, v_2, \ldots, v_r)$, which we call *fundamental*. This cycle is represented by a $(0, 1)$-sequence of length n corresponding to a row of A.
Notice that the vertices $1_n = (1, 1, \ldots, 1)$ and $0_n = (0, 0, \ldots, 0)$ determine fundamental cycles of length 1.

We may observe that the fundamental cycles are simply the equivalence classes of the rotation; then it is immediate to prove the following lemma.

Lemma 1. *The set of vertices of B_n can be partitioned into disjoint fundamental cycles.*

A consequence is that the set of fundamental cycles turns out to be the quotient set of $V(B_n)$ with respect to rotation of the n-sequences representing the vertices of B_n.
For example the vertices of B_3 may be partitioned into the fundamental cycles $(0, 0, 0)$, $(1, 1, 1)(0, 1, 0)$, $(1, 1, 0)$, which represent the partitions $\emptyset$, $(1, 1, 1)$, (3), $(2, 1)$.

3. Orbital representatives

In this section we characterize partitions that in each orbit have the least numerical value. First we define a minimal $(0, 1)$-sequence and a minimal partition.

Definition 1. A $(0, 1)$-sequence $\mathbf{a}$ is said *minimal* if

$$n(\mathbf{a}) = N(\mathbf{a}).$$

Thus an ordered partition λ is said *minimal* if

$$n(\lambda) = N(\lambda)$$

It is easy to prove the following characterization of minimal ordered partitions [2].

Theorem 1. *An ordered partition $\lambda = (k_1, k_2, \ldots, k_m)$ of n is minimal if and only if following properties hold:*

1. $$k_1 \ge k_i, \quad 2 \le i \le m;$$
2. *If, for suitable indices h and l,*

$$k_1 = k_{1+h}, k_2 = k_{2+h}, \ldots, k_l = k_{l+h}$$

and

$$k_{l+1} \neq k_{h+l+1},$$

then

$$k_{l+1} > k_{h+l+1},$$

where indices are mod m.

For example (3, 2, 3, 1, 1) is a minimal ordered partition of 10, while (3, 1, 1, 3, 2) is not minimal.
In the following we denote by $fp(\lambda)$ the first part of an ordered partition λ.
Let α be a minimal partition of n represented by the sequence $(b_1, b_2, \ldots, b_n)$. We define the complement of α to be the minimal partition $\overline{\alpha}$ represented by a suitable rotation of the sequence $(\bar{b}_n, \bar{b}_{n-1}, \ldots, \bar{b}_1)$, in order to satisfy Theorem 1.

For example the complement of (4, 1, 2) is (3, 1, 1, 2).

4. Graph associated to a binary de Bruijn digraph

In relation to a binary de Bruijn digraph B_n we may associate the graph $G(B_n) = (V, E)$, defined in the Introduction. Now we notice that V coincides with the set of fundamental cycles in which $\mathbb{Z}_2^n$ may be partitioned and two cycles $C_1, C_2 \in V$ are adjacent when there is a vertex $u \in V(C_1)$ and a vertex $v \in V(C_2)$ such that (u, v) is an arc of B_n.

In a recent combinatorial conference J. Mykkeltveit posed to the first author the following problem:

Does $G(B_n)$ contain a perfect matching?

The interest to this problem arises from its connection to the study of the periods of non-linear recurrences. In this paper we answer in the positive in the case of n odd, while the case of n even remains unsolved.

Proposition 1. *Let $n \geq 2$; two vertices of $G(B_n) \setminus \{(0, \ldots, 0)\}$ are adjacent if and only if the corresponding partitions of n satisfy the condition that one is obtained from the other one by replacing two consecutive parts with their sum.*

Proof. Let C_1 and C_2 two fundamental disjoint cycles of B_n.
Assume that they are adjacent in G. This implies there is a vertex $u \in C_1$ and a vertex $v \in C_2$ such that (u, v) is an arc of B_n.
If $u = (x_1, x_2, \ldots, x_n)$ and $v = (y_1, y_2, \ldots, y_n)$, then

$$x_2, x_3, \ldots, x_n = y_1, y_2, \ldots, y_{n-1}.$$

It follows that $x_1 \neq y_n$, otherwise v is obtained by shifting u and the cycles would coincide.
Assume, without loss of generality, $x_1 = 1$ and $y_n = 0$. Thus if $m \geq 1$ is the number of ones of v, then $m + 1$ is the number of ones of u. This implies that v and u may be partitioned into m and $m + 1$ parts respectively, whose sequences coincide, but one part of v which is the sum of two consecutive parts in u.

Conversely, let α and β be ordered partitions satisfying previous condition. Without loss of generality we may represent $\alpha = (a_1, a_2, \ldots, a_h)$ and $\beta = (a_1 + a_2, a_3, \ldots, a_h)$. This implies that the (0, 1)-sequences $s(\alpha)$ and $s(\beta)$ differ in only one position. Up to a suitable rotation, we may denote $s(\alpha) = (x_1, x_2, \ldots, x_n)$ and $s(\beta) = (x_2, x_3, \ldots, x_n, y)$, where $x_1 \neq y$; thus the cycles corresponding to $s(\alpha)$ and $s(\beta)$ are adjacent. □

The following corollaries are an immediate consequence of Proposition 1.

Corollary 1. *A necessary condition to the adjacency of two vertices of G is that the numbers of parts of the corresponding partitions differ exactly by* 1.

Corollary 2. *Two vertices of G are adjacent if and only if up to suitable rotations the corresponding* (0, 1)*-sequences differ in exactly one position.*

This corollary implies that the cycle (0, . . . , 0) is adjacent to (0, . . . , 0, 1); in other words the partition ∅ is adjacent to the partition (n).

A vertex of G is said *even* or *odd* according to the condition that the corresponding partition has an even or an odd number of parts respectively.

Lemma 2. *Two vertices of G, having the same parity, are not adjacent.*

Proof. If the partitions have the same parity, then the numbers of parts is the same or differ by a positive even number. By previous Corollary 1 they are not adjacent. □

Proposition 2. *Let $n > 1$; the graph $G(B_n)$ is bipartite and connected.*

Proof. Because fundamental cycles correspond to minimal ordered partitions of n and partitions of the same parity are not adjacent, it follows that $G(B_n)$ is bipartite.

Let $\alpha = (a_1, a_2, \ldots, a_r)$ and $\beta = (b_1, b_2, \ldots, b_s)$ be two minimal partitions of n. Comparing a_1 and b_1 we have three cases:

1. $a_1 > b_1$;
2. $a_1 < b_1$;
3. $a_1 = b_1$.

In the first case we determine the partition :

$$\gamma_1 = (a_1 - 1, a_2, \dots, a_r, 1)$$

which is adjacent to α. If γ_1 is not ordered according to Theorem 1, we replace it by a suitable rotated partition γ_1'. Notice that γ_1' is still adjacent to α and the first part is $\geq b_1$. If it is greater than b_1 we repeat the procedure until to obtain a partition having the first part equal to b_1. Then we compare γ_1 (or γ_1') and β.

In the second case, by a similar way, we determine the partition

$$\delta_1 = (b_1 - 1, b_2, \dots, b_s, 1),$$

which turns out to be adjacent to β; then we consider α and δ_1 (or δ_1').

If $a_1 = b_1$, then we have to determine the first part of α which is different from the corresponding part of β. Let us assume that $\alpha = (a_1, \dots, a_t, a_{t+1}, \dots, a_r)$ and $\beta = (a_1, \dots, a_t, b_{t+1}, \dots, b_s)$ and $a_{t+1} \neq b_{t+1}$. Without loss of generality we may assume $a_{t+1} > b_{t+1}$.
Then we may consider $\lambda_1 = (a_1, \dots, a_t, (a_{t+1} - 1), 1, a_{t+2}, \dots, a_r)$. Then we have to compare λ_1 (or λ_1') and β.
In any case now we have to consider two partitions whose first part is the same.

By iterating the procedure we arrive to consider two partitions γ_j and δ_k whose parts starting from left coincide but last part of one, which turns out to be the sum of last two parts of the other one.
Thus these partitions are adjacent; then the sequence $(\alpha, \gamma_1, \dots, \gamma_j, \delta_k, \dots, \delta_1, \beta)$ is a path of $G(B_n)$ connecting the vertices α and β. □

For example, let $\alpha = (4, 2, 3)$ and $\beta = (3, 3, 1, 1, 1)$; then $\gamma_1 = (3, 2, 3, 1)$, $\delta_1 = (3, 2, 1, 1, 1, 1)$ and then $\gamma_2 = (3, 2, 2, 1, 1)$. Thus we have the sequence $(\alpha, \gamma_1, \gamma_2, \delta_1, \beta)$.

5. Perfect matching

Let V_e and V_o the sets of even and odd vertices of G respectively.

Proposition 3. *Let $n > 1$ be odd; the sets V_e and V_o have the same cardinality.*

Proof. Let $n = 2t + 1$, $t \geq 1$. Denote by P_k the set of ordered partitions of n in k parts and $\bar{P_k}$ the set of complementary partitions of P_k. It is easy to see that $\bar{P_k} = P_{n-k}$. Moreover if k is even, then $n - k$ is odd and conversely. In particular $\bar{V_e} = V_o$; then the result holds. □

Recall that a subset of edges of a graph G is called a *matching* if no two edges have a common endvertex.
Let M be a matching of G. An M- *alternating* path is a path whose edges alternate between edges not in M and edges in M.
A vertex v of G is *saturated* by M if it is the endvertex of an edge of M; otherwise the vertex v is *unsaturated* by M.
Two vertices $v, w \in G$ are said *coupled* by M if (v, w) is an edge of M.
An M- *augmenting* path is an M- alternating path that begins and ends with M- unsaturated vertices.
A *perfect* matching of the bipartite graph $G(B_n)$ is one that saturates all vertices of the graph.

Proposition 4. *Let M be a non perfect matching of $G(B_n)$; then $G(B_n)$ contains an M- augmenting path.*

Proof. Let M be a not perfect matching of G; then there are at least an even vertex and an odd vertex of G not saturated by M.
Let $\alpha = (a_1, a_2, \dots, a_r) \in V_e$ and $\beta = (b_1, b_2, \dots, b_s) \in V_0$ be vertices not saturated by M.
If α is adjacent to β, then the edge (α, β) may be considered an M- augmenting path.
If they are not adjacent, from the comparison of a_1 and b_1, we have three cases to consider:

1. $$a_1 > b_1$$
2. $$a_1 < b_1$$
3. $$a_1 = b_1.$$

1. Assume that $a_1 > b_1$. Then consider the vertex $\gamma_1 = (a_1 - 1, a_2, \dots, a_r, 1)$. If $a_1 - 1$ is greater than all the remaining elements, we have that the sequence remains minimal; otherwise we possibly replace it by a suitable rotated sequence. Let $(c_1, c_2, \dots, c_{r+1})$ be its new minimal form, corresponding to the vertex γ_1. Notice that by the assumption $c_1 \leq a_1$.
If γ_1 is not saturated by M, then the edge (α, γ_1) is an M- augmenting path.
Otherwise denote $\gamma_1' = (c_1', c_2', \dots, c_{r'}')$ the vertex coupled with γ_1 by M, where $r' \in \{r, r+2\}$.

2. In the case of $a_1 < b_1$, we start from β and repeat the same previous procedure, thus obtaining either an edge (β, δ_1) to be add to M or a path $(\beta, \delta_1, \delta_1')$ where the second edge belongs to M.
3. If $a_1 = b_1$, then we have to determine the first part of α which is different from the corresponding part of β. Let us assume that $\alpha = (a_1, \dots, a_t, a_{t+1}, \dots, a_r)$ and $\beta = (a_1, \dots, a_t, b_{t+1}, \dots, b_s)$ and $a_{t+1} \neq b_{t+1}$.
   If $a_{t+1} > b_{t+1}$, then we may consider $\lambda_1 = (a_1, \dots, a_t, (a_{t+1} - 1), 1, a_{t+2}, \dots, a_r)$. Again we have either an edge (α, λ_1) to be add to M or a path $(\alpha, \lambda_1, \lambda_1')$ whose second edge belongs to M. A similar situation holds in the case of $a_{t+1} < b_{t+1}$.
   In any case we determine either an edge to be add to M or a path of length 2, starting from one of the vertices α or β, whose second edge belongs to M.
   We may continue in a similar way.
   If we consider the path $(\alpha, \gamma_1, \gamma_1')$, we have to compare the first part of γ' with the first part of β and then we arrive to determine a path $(\alpha, \gamma_1, \gamma_1', \dots, \gamma_j, \gamma_j')$, where $fp(\gamma_h') > b_1$, for $1 \leq h < j$, and $fp(\gamma_j') \leq b_1$.
   In the case in which $fp(\gamma_j') < b_1$, we start from β and determine a path $(\beta, \delta_1, \delta_1', \dots, \delta_i, \delta_i')$, where $fp(\delta_q') > fp(\gamma_j')$ for every $1 \leq q < i$ and $fp(\delta_i') \leq fp(\gamma_j')$.
   If $fp(\gamma_j') = b_1$, we continue by considering the first part, which is different form the corresponding part of β.
   Notice that if, in relation to a vertex a vertex $\gamma_h' = (d_1, d_2, \dots, t_t)$, the vertex $\gamma_{h+1} = (d_1 - 1, d_2, \dots, d_t, 1)$ coincides with a vertex yet used, then we replace γ_{h+1} by the vertex $(d_1 - 1, 1, d_2, \dots, d_t)$ or by a vertex obtained from γ_h' by replacing one part $k > 1$ by two consecutive parts whose sum is equal to k.
   Because the graph is connected, at last we obtain either a path which starts from α or β and ends by a non saturated vertex. Otherwise we have two different paths $(\alpha, \dots, w)$ and $(\beta, \dots, z)$, where w and z are adjacent and are coupled with the preceding vertex in the path.
   This condition allows us to determine the M- augmenting path $(\alpha, \dots, w, z, \dots, \beta)$, thus completing the proof of the theorem. □

Theorem 2. *Let n be odd; the graph $G(B_n)$ contains a perfect matching.*

Proof. Assume that G contains a matching M which is not perfect. By Proposition 4 it is possible to determine an M-augmenting path; it is well known that it is always possible to determine a new matching M' such that

$$|M'| = |M| + 1.$$

By iterating the process the result follows. □

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